

# Fragility of String Orders

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One-dimensional gapped systems are often characterized by a 'hidden' non-local order parameter, the so-called string order. Due to the gap, thermodynamic properties are robust against a weak higher-dimensional coupling between such chains or ladders. To the contrary, we find that the string order is not stable and decays for arbitrary weak inter-chain or inter-ladder coupling. We investigate the vanishing of the order for three different systems: spin-one Haldane chains, band insulators, and the transverse field Ising model. Using perturbation theory and bosonization, we show that the fragility of the string order arises from non-local commutation relations between the non-local order parameter and the perturbation.

PACS numbers: 71.10.Fd, 71.10.Hf, 75.10.Lp, 75.10.Pq

Since the pioneering work of Landau<sup>1</sup> that led to the theory of 2nd order phase transitions, the concept of order parameter has become one of the main paradigms in condensed matter physics. The central observation is that when a symmetry is spontaneously broken and long range order appears into the system, certain correlation functions of operators<sup>2</sup>  $A(x)$  do not decay with distance,

$$\lim_{|x-y| \rightarrow \infty} \langle A(\mathbf{x})A(\mathbf{y}) \rangle = c \neq 0. \quad (1)$$

and it is possible to define an order parameter as  $\langle A \rangle = \sqrt{c}$ . For example, it is known that a classical three-dimensional ferromagnet undergoes a phase transition, becoming ordered (all the spins point in the same direction) below a certain critical temperature. In this case, the order parameter can be identified with the spontaneous magnetization  $\langle \mathbf{S}(x) \rangle = m$  and the spin-spin correlation function  $\langle \mathbf{S}(x)\mathbf{S}(y) \rangle = m^2$  is finite also in the limit  $|x-y| \rightarrow \infty$ . This simple way to describe and discriminate the different phases of matter has been successfully applied to a plethora of different systems both at finite and zero temperature. Classical and quantum magnetism (with many possible order parameters: uniform and staggered magnetization, dimerization...), superconductivity (Cooper pair amplitude), superfluidity (condensate amplitude) are only few of the many successful applications of the Landau paradigm.

In the context of low dimensional quantum systems, it has been also useful, under some circumstances, to identify 'non-local' order parameters which cannot be written in the form (1). Typically, this is the case of some gapped one-dimensional Hamiltonians with no local symmetries that are spontaneously broken and with any two-point correlation function that decays exponentially. In such systems an hidden long range order can nevertheless be present and this is encoded in the long distance behavior of certain non-local operators. Notice that a 'non-local' order parameter is not directly accessible to any experimental probe but can be equally used to mark theoretically the boundaries of the different phases. In this context, an important example is the so-called "string-

order" defined by the non-local correlation function

$$\lim_{|x-y| \rightarrow \infty} \langle A(\mathbf{x}) \left( \prod_{\mathbf{z} \in S_{\mathbf{x},\mathbf{y}}} B(\mathbf{z}) \right) A(\mathbf{y}) \rangle \neq 0, \quad (2)$$

where the operator  $\prod_{\mathbf{z} \in S_{\mathbf{x},\mathbf{y}}} B(\mathbf{z})$  acts on a line (the string  $S_{\mathbf{x},\mathbf{y}}$ ) connecting the points  $\mathbf{x}$  and  $\mathbf{y}$ . One dimensional examples of this family (discussed further below) includes the spin-1 chain<sup>3</sup> and spin-1/2 ladders<sup>4</sup>, the transverse-field quantum Ising chain (dual order), band- and Mott-Fermionic insulators<sup>5</sup> and, among Bosonic systems, some parameter regime of the Bose-Hubbard Hamiltonian<sup>6</sup>.

Another important sub-class of these more exotic types of systems is characterized by the so-called 'topological order' (e.g. fractional quantum Hall fluids). This concept can be defined by the ground state degeneracy on non-trivial manifolds<sup>7,8,9</sup>. Here we will only consider systems, as listed above, with unique ground states on a torus and no topological order.

In this paper, we want to address the following questions: Is the string order stable against small perturbations? And can it be generalized from one- to higher-dimensional systems? More precisely, we will investigate weakly coupled two- or three-dimensional arrays of one-dimensional systems. In the absence of the higher-dimensional coupling  $\lambda_{\perp}$  they are characterized by various types of string order and a finite gap in the spectrum (but no topological order). Due to this gap, a sufficiently small  $\lambda_{\perp}$  will never induce any thermodynamic phase transition. Therefore, one might naively expect that also the order parameter is robust against such a small perturbation. While this is correct for *local* order parameters, we will show that generically it does not hold for the non-local string order.

In the following, we will first consider, as a prototypical example, the disordered phase of the quantum Ising chain in transverse field characterized by a hidden string order parameter. Here the divergence of the perturbation theory indicates that string order is destroyed by arbitrary small higher-dimensional coupling  $\lambda_{\perp}$ . We connect this result – that also holds for the case of the spin-1 Haldane chain – to the band-insulators case where an exact calcu-

lation of the string order-parameter is possible, proving the absence of string order for any finite  $\lambda_\perp$ . Using the language of bosonization, we identify the general mechanism destabilizing non-local order. In contrast, local order (e.g. a charge density wave) remains stable.

*Transverse field Ising model:* As a starting point for our discussion, we introduce the Hamiltonian of the quantum Ising chain

$$H = - \sum_i J \sigma_i^z \sigma_{i+1}^z - B \sigma_i^x, \quad (3)$$

describing a quantum magnet in a transverse field. We will always consider  $T = 0$  and the ferromagnetic case  $J > 0$  ( $J < 0$  leads nevertheless to completely equivalent physics). The transverse-field Ising chain is a textbook example in the context of quantum criticality. Upon increasing the tuning parameter  $B/J$ , the ground state experiences a quantum phase transition from a magnetic to a paramagnetic state and the model can be solved exactly with the use of the standard Fermionic representation for a quantum spin<sup>10</sup>. Interestingly, there is a subtle way to identify the critical value of  $B/J$  that exploits a hidden non-local property. In perfect analogy with the Kramers-Wannier<sup>11</sup> duality transformation for the classical two-dimensional Ising model, one can introduce the following mapping<sup>12</sup>

$$\begin{aligned} \hat{\mu}_i^x &= \hat{\sigma}_{i+1}^z \hat{\sigma}_i^z \\ \hat{\mu}_i^z &= \prod_{m \leq i} \sigma_m^x, \end{aligned} \quad (4)$$

that preserves the  $SU(2)$  algebra and transforms Eq. (3) into

$$H = \sum_i B \mu_i^z \mu_{i+1}^z + J \mu_i^x. \quad (5)$$

i.e. one obtains the same Hamiltonian as in (3) when replacing  $J \leftrightarrow B$ ,  $\mu_i^\alpha \leftrightarrow \sigma_i^\alpha$ . Therefore the quantum critical point has to be located at the self-dual point  $J/B = 1$ .

As in the ferromagnetic phase  $B < J$ ,  $\langle \sigma^z \rangle$  is finite, one finds in the disordered phase,  $B > J$ , that  $\langle \mu^z \rangle$  is finite, or more precisely

$$\lim_{(j-i) \rightarrow \infty} \langle \mu_i^z \mu_j^z \rangle = \lim_{(j-i) \rightarrow \infty} \left\langle \prod_{i < m \leq j} \sigma_m^x \right\rangle > 0 \quad (6)$$

Therefore the disordered phase is characterized by non-local string order (2).

If we now take a pair of these quantum Ising chains, we can investigate the fate of the non-local order when a weak inter-chain coupling  $J_\perp$  is present. For simplicity, we set  $J = 0$  and consider the Hamiltonian

$$H = \sum_i B \sigma_{1,i}^x + B \sigma_{2,i}^x + J_\perp \sigma_{1,i}^z \sigma_{2,i}^z. \quad (7)$$

As we have set  $J = 0$ , the system is a sum of independent two-site Hamiltonians and is trivial to calculate the string

order exactly

$$\left\langle \prod_{i < m \leq j} \sigma_{1,m}^x \right\rangle = \langle \sigma_{1,1}^x \rangle^{|j-i|} \approx e^{-\frac{J_\perp^2}{8B^2} |j-i|} \quad (8)$$

where we used that  $\langle \sigma^x \rangle = 2B/\sqrt{4B^2 + J_\perp^2} \approx 1 - J_\perp^2/(8B^2)$  to leading order in  $J_\perp/B$ . The string order, being a product of factors strictly less than 1, decays exponentially for any  $J_\perp \neq 0$ .

So far, the vanishing of the string order can be an artifact due to the absence of interactions. To see that a finite  $J < B$  cannot stabilize the order, it is instructive to repeat the calculation in the dual variables. In this language, the order parameter is now local but the coupling  $J_\perp$  induces a non-local term in the Hamiltonian

$$H = \sum_i B \mu_{1,i}^z \mu_{1,i+1}^z + B \mu_{2,i}^z \mu_{2,i+1}^z + J_\perp \left( \prod_{m \leq i} \mu_m^x \right) \left( \prod_{k \leq i} \mu_k^x \right). \quad (9)$$

With standard text-book techniques, we expand the S-matrix to second order in  $J_\perp$  and obtain for the two-point correlation function

$$\langle \mu_{1,i}^z \mu_{1,j}^z \rangle \approx 1 - \frac{J_\perp^2 |i-j|}{8B^2} \quad (10)$$

consistent with (8). Formally, the divergence with  $|i-j|$  arises from the term

$$\int_0^\infty dt_1 \int_{-\infty}^0 dt_2 \sum_{k,l} \langle \prod_{m \leq k} \mu_m^x e^{-iH_0 t_1} \mu_i^z \mu_j^z e^{+iH_0 t_2} \prod_{s \leq l} \mu_s^x \rangle_0 \quad (11)$$

as a consequence of the non-local commutation relation between the order parameter and the perturbation

$$\left[ \mu_i^z, \prod_{m \leq l} \mu_m^x \right] = [\mu_i^z, \sigma_l^z] = i \left( \mu_i^y \prod_{m \leq l, m \neq i} \mu_m^x \right) \Theta(l-i). \quad (12)$$

Physically,  $\prod_{m \leq l} \mu_m^x$  describes the creation of a domain wall. As any domain wall created between the points  $i$  and  $j$  destroys the correlations of  $\mu_{1,i}^z$  and  $\mu_{1,j}^z$ , the perturbation theory diverges linearly in  $|i-j|$ . From the perturbative argument, it is easy to see that a finite  $J < B$  does not change this picture qualitatively for  $|i-j|$  large compared to the correlation length  $\xi \sim (B-J)^{-1}$ . In fact, the interaction term  $\sum J \sigma_i^z \sigma_{i+1}^z = \sum J \mu_i^x$  has local commutation relations with the order parameter and produces only regular corrections to the perturbation series. These additional non-singular terms cannot compensate for the linear divergence induced by the inter-chain coupling.

Even though the long range order vanishes, thermodynamics is unaffected by  $J_\perp$  (up to a small renormalization of the gap): obviously no quantum phase transition is induced. Only the order parameter is sensitive to the non-locality of the perturbations. In the language of the original variables  $\sigma_i^\alpha$ , in contrast, the perturbation was

local, the order parameter non-local, and the same type of divergence arises again due to their non-local commutation relations (12).

Finally, notice that in the ferromagnetic phase,  $J > B$ , the local magnetic order is stabilized rather than suppressed by  $J_\perp$ . While for  $B > J$ ,  $J_\perp$  induces a finite density of virtual ‘dual’ domain wall fluctuations, the ‘physical’ domain walls are suppressed for  $B < J$  by  $J_\perp$ . More precisely, they are confined as the energy of a pair of domain walls with separation  $|i - j|$  is proportional to  $|i - j|J_\perp^2/\Delta$  in the ferromagnetic phase<sup>13</sup>.

*Spin 1 chain:* A second major type of non-local order that we want to consider is the so-called string order. This was introduced first in 1989 by Den Nijs and Rommelse<sup>3</sup> briefly after that Haldane conjectured the existence of a gap in the spin-1 antiferromagnetic chain as the hidden order of the Haldane phase. They observed that, even though true Néel order is absent, the ground state has still a form of long range order characterizing the entanglement of the spins: any site with  $S^z = \pm 1$  is always followed by another with  $S^z = \mp 1$ , separated from the first by a string of  $S^z = 0$  of arbitrary length. This implies that the string order parameter

$$SO_{\text{chain}}(i - j) = \left\langle S_i^z \exp\left(i\pi \sum_{l=i+1}^{j-1} S_l^z\right) S_j^z \right\rangle \quad (13)$$

is always finite for  $|i - j| \rightarrow \infty$  in the Haldane phase.

Kennedy and Tasaki<sup>14,15</sup> observed that this non-local order can be understood from the non-local mapping

$$\begin{aligned} \tilde{S}_j^x &= S_j^x \exp\left(i\pi \sum_{k=j+1}^L S_k^x\right) \\ \tilde{S}_j^y &= \exp\left(i\pi \sum_{k=1}^{j-1} S_k^z\right) S_j^y \exp\left(i\pi \sum_{k=j+1}^L S_k^x\right) \\ \tilde{S}_j^z &= \exp\left(i\pi \sum_{k=1}^{j-1} S_k^z\right) S_j^z \end{aligned} \quad (14)$$

as it maps the Heisenberg Hamiltonian with open boundary conditions to an effective *local* ferromagnetic Hamiltonian with a manifest  $Z_2 \times Z_2$  symmetry that is fully broken in the ground-state (the origin of the 4-fold ground state degeneracy for such boundaries is well understood in the AKLT picture<sup>16</sup> and comes as a consequence of an effective spin 1/2 localized at each boundary). The same transformation, applied to the inter-chain coupling, introduces in the Hamiltonian domain-wall creation operators similar to the ones of Eq. (9). Therefore the absence of the string order in the presence of a finite  $J_\perp$  can be shown as above. Indeed, the vanishing of the string order for two coupled spin-1 chains was observed numerically by Todo *et al.*<sup>17</sup> using the quantum Monte Carlo method.

*Ladders:* Similar to the spin-1 chain, also gapped spin-1/2 ladders are characterized by a non-local string order<sup>4</sup>. In this case, one can distinguish between two different

types of string orders,  $SO_{\text{odd}}$  and  $SO_{\text{even}}$  (see Ref. [4])

$$SO_{\text{odd}}(i - j) = - \left\langle (S_{1,i}^z + S_{2,i}^z) \exp\left(i\pi \sum_{l=i+1}^{j-1} S_{1,l}^z + S_{2,l}^z\right) (S_{1,j}^z + S_{2,j}^z) \right\rangle \quad (15)$$

$$SO_{\text{even}}(i - j) = - \left\langle (S_{1,i+1}^z + S_{2,i}^z) \exp\left(i\pi \sum_{l=i+1}^{j-1} S_{1,l+1}^z + S_{2,l}^z\right) (S_{1,j+1}^z + S_{2,j}^z) \right\rangle. \quad (16)$$

where the spin-1 of Eq. (13) is replaced by the sum of two spin-1/2 operators on either the vertical or on the diagonal rung.

In a recent paper<sup>5</sup>, we have shown that the ground state of gapped spin-1/2 ladders (and of the spin-1 chain) is adiabatically connected to the one of an ordinary, non-interacting band insulator. In Ref. [5], we introduced the following family of ladder Hamiltonians

$$\begin{aligned} H &= \sum_{i,\alpha,\sigma} t_\alpha a_{\alpha,i,\sigma}^\dagger a_{\alpha,i+1,\sigma} + h.c. - \frac{U}{2} n_{\alpha,i,\sigma} \\ &+ \sum_{i,\sigma} t_R a_{1,i,\sigma}^\dagger a_{2,i,\sigma} + t_D a_{1,i+1,\sigma}^\dagger a_{2,i,\sigma} + h.c. \\ &+ U \sum_{i,\alpha} n_{\alpha,i,\uparrow} n_{\alpha,i,\downarrow} + J_R \sum_i \mathbf{S}_{1,i} \cdot \mathbf{S}_{2,i} \end{aligned} \quad (17)$$

(where  $\alpha = 1, 2$  and  $\sigma = \pm$  are the row and spin indices) whose phase diagram includes both Mott ( $U, J_R \gg t_i$ ) and band insulating phases ( $U = J_R = 0$ ). Remarkably, the string order (15,16) turns out to be finite also for the band insulator.

In the non-interacting case ( $U = J_R = 0$ ), the string order can be calculated exactly<sup>5</sup> even for an array of such one-dimensional insulators. As shown in Fig. 1, in the presence of an arbitrarily weak inter-ladder coupling

$$H_\perp = t_\perp \sum_{i,l,\alpha,\sigma} a_{2,i,l,\sigma}^\dagger a_{1,i,l+1,\sigma} + h.c. \quad (18)$$

(where the extra index  $l$  labels the ladders), the string order decays exponentially

$$SO \sim e^{-\alpha|i-j|}, \quad \alpha \propto t_\perp^2 \quad (19)$$

as in Eq. (8). The prefactor  $\alpha$  is quadratic in the inter-ladder hopping  $t_\perp$  (see inset of Fig. 1).

*Bosonization:* To identify the precise origin of the fragility of string order, we now analyze a generic one-dimensional gapped phase using the language of (Abelian) bosonization. Within this approach, the relevant degrees of freedom are described by the fields  $\Phi$  and  $\Theta$  obeying the non-local commutation relation

$$[\Phi(x), \Theta(x')] = i\theta(x - x') \quad (20)$$

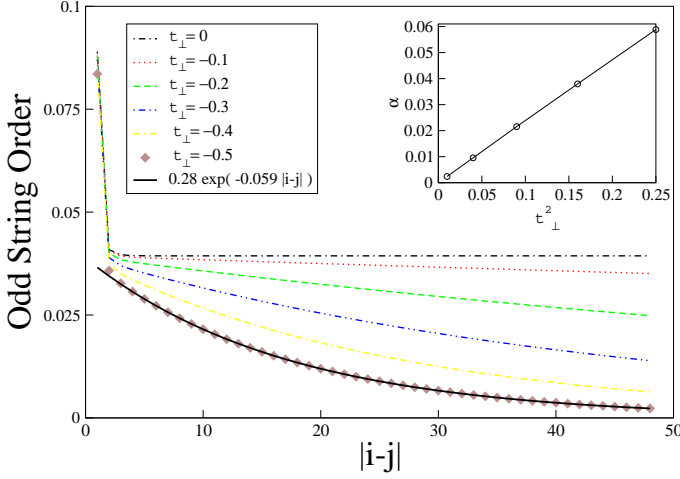


FIG. 1: (Color online) The odd string order for a two-dimensional array of weakly coupled non-interacting band insulators decays as  $SO_{\text{odd}} \approx e^{-\alpha|i-j|}$ . For  $t_{\perp} = -0.5$  we also show the exponential fit. Inset:  $\alpha$  as a function of  $t_{\perp}^2$ . Both plots are done for the set of parameters  $t_1 = 0$ ,  $t_2 = -0.6$ ,  $t_R = -1.5$ ,  $t_D = -2$  of the model defined in Eqns. (17, 18).

where  $\theta(x)$  is the usual  $\theta$ -function. Gapped phases typically arise when one of the Bosonic fields, e.g.  $\Phi$ , is locked by a 'relevant' non-linear interaction

$$H = \int dx \frac{v_F}{2} \left[ \frac{1}{K} (\partial_x \Theta)^2 + K (\partial_x \Phi)^2 \right] + g \cos(\chi \Phi) \quad (21)$$

where  $K$  is the Luttinger liquid parameter and  $g$  and  $\chi$  parameterize the most relevant perturbation of the Luttinger liquid fixed point. The Hamiltonian has two symmetries: First, it is invariant under the shift  $\Theta \rightarrow \Theta + c$  by an arbitrary constant  $c$  reflecting the conservation of the 'charge'  $\int \partial_x \Phi$  (typically either the total  $S^z$  or the total number of electrons). Second, the field  $\Phi$  is invariant under a shift  $\Phi \rightarrow \Phi + c_{\phi}$  by a *fixed* number  $c_{\phi}$ , reflecting charge quantization and giving rise to the constraint  $\chi = n2\pi/c_{\phi}$  with integer  $n$ . For example, in the case of a non-interacting band-insulator  $c_{\phi} = 2\sqrt{\pi}$ ,  $\chi = \sqrt{4\pi}$ ,  $K = 1$  and  $g$  is proportional to the  $2k_F$  (where  $k_F$  is the Fermi momentum) component of the periodic potential.

In this language the origin of the string order is easy to understand. As the field  $\Phi$  is locked in one of the minima of the cosine term, any correlation function of the form

$$O_{\gamma}(x-y) = \langle e^{i\gamma\Phi(x)} e^{-i\gamma\Phi(y)} \rangle \quad (22)$$

does not decay for  $|x-y| \rightarrow \infty$  and has a finite value

$$\lim_{|x-y| \rightarrow \infty} O_{\gamma}(x-y) = \mu_{SO}^2. \quad (23)$$

For example, a 'string operator' as in Eq. (15) is bosonized in the continuum limit by<sup>4</sup>

$$\exp\left(i\bar{\gamma} \sum_{l=i+1}^{j-1} S_{1,l}^z + S_{2,l}^z\right) \approx \exp\left[i \frac{\bar{\gamma}}{\sqrt{2\pi}} (\Phi_s(x) - \Phi_s(y))\right] \quad (24)$$

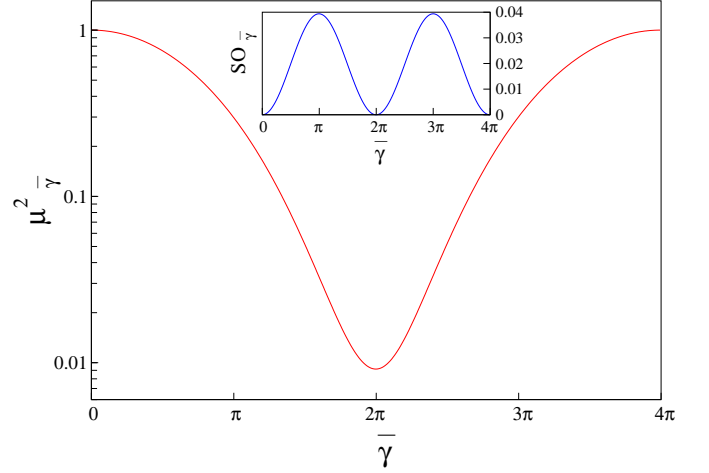


FIG. 2: (Color online) The string order parameter without including the boundary spins defined as  $\mu_{\bar{\gamma}}^2 = \lim_{j \rightarrow \infty} \langle \exp(i\bar{\gamma} \sum_{l=1}^{j-1} S_{1,l}^z + S_{2,l}^z) \rangle$  for a one-dimensional band insulator on a ladder (for the value of the parameters, see Fig. 1) as a function of  $\bar{\gamma}$ . By construction  $\mu_{\bar{\gamma}} = 1$  for  $\bar{\gamma} = 4\pi n$ . Inset: String order  $SO_{\bar{\gamma}} = \lim_{j \rightarrow \infty} \langle (S_{1,j}^z + S_{2,j}^z) \exp(i\bar{\gamma} \sum_{l=1}^{j-1} S_{1,l}^z + S_{2,l}^z) (S_{1,0}^z + S_{2,0}^z) \rangle$  including the boundary spins. Notice that at  $\bar{\gamma} = 2\pi$  the  $SO_{2\pi}$  is exactly zero while the  $\mu_{2\pi}^2$  has a small but finite value. This arises because the operator  $e^{i2\pi S_z}$  is independent of the spin configuration.

where  $\partial_x \Phi_s$  is proportional to the total spin density. Note that in Eqs. (13), (16) and (15) besides the string operator, also extra boundary spins at site  $i$  and  $j$  have been included in the definition of the string order. However, these terms are not essential, as discussed in the caption of Fig. 2, and string order is present as long as  $O_{\gamma}(x-y)$  is finite for  $|x-y| \rightarrow \infty$ .

We can make use of this formalism to understand the consequence of an inter-chain coupling. In the presence of such a perturbation, the total 'charge'  $\int \partial_x \Phi$  on a *single* chain is not conserved any more. This is reflected in the appearance of the dual field  $\Theta$  in the Hamiltonian (more specifically, of the exponential  $e^{i\beta\Theta(x)}$ ). For instance, for chains coupled by single-electron hopping one obtains

$$H_{\perp} = t_{\perp} \sum_{l,\sigma=\uparrow/\downarrow} \Psi_{l,\sigma}^{\dagger} \Psi_{l+1,\sigma} + h.c. \\ \sim \frac{1}{2\pi a} \sum e^{i\sqrt{\frac{\pi}{2}}(\Phi_{c,l} \pm \Phi_{s,l} + \Theta_{c,l} \pm \Theta_{s,l})} \\ \times e^{-i\sqrt{\frac{\pi}{2}}(\Phi_{c,l+1} \pm \Phi_{s,l+1} + \Theta_{c,l+1} \pm \Theta_{s,l+1})} + \dots \quad (25)$$

where the summation index  $l$  spans the different ladders and we introduced the spin and charge Bosonic fields (and their duals). In the last equality only the  $\Psi_{L,l,\sigma}^{\dagger} \Psi_{L,l+1,\sigma}$  component is shown.

According to the commutation relations (20), the operator  $e^{i\beta\Theta(x)}$  increases the  $\Phi(x')$  field by  $\beta$  for  $x' < x$ . In a semi-classical picture,  $e^{i\beta\Theta(x)}$  therefore creates a domain

wall at  $x$  by shifting  $\Phi$  for  $x' < x$  from one minimum of the cosine to another (note that  $\beta$  is always an integer multiple of  $2\pi/\chi$  as a consequence of charge quantization).

In analogy to the calculation for the transverse field Ising model, Eq. (11), we can now proceed by calculating the corrections to the string order parameter perturbatively in  $H_\perp = \int dx h_\perp(x)$ . Here we assume that the string order is calculated on the chain with index  $l = 0$  and is defined in terms of the spin field  $\Phi_{s,0}$ ,  $O_\gamma(x - y) = \langle e^{i\gamma(\Phi_{s,0}(x,0) - \Phi_{s,0}(y,0))} \rangle$ . To second order we obtain

$$\frac{1}{2} \int_{-\infty}^{+\infty} dx_1 dt_1 \int_{-\infty}^{+\infty} dx_2 dt_2 \langle T e^{i\gamma(\Phi_{s,0}(x,0) - \Phi_{s,0}(y,0))} \times h_\perp(x_1, t_1) h_\perp(x_2, t_2) \rangle_c \quad (26)$$

Here,  $\langle \dots \rangle_c$  denotes the connected part of the correlation function (i.e.  $\langle e^{i\gamma(\Phi_{s,0}(x) - \Phi_{s,0}(y))} \rangle \langle h_\perp h_\perp \rangle$  has been subtracted) and the time-ordering  $T$  splits the integral in the four different contributions ( $t_1 < 0 < t_2$ ), ( $t_2 < 0 < t_1$ ) ( $t_1, t_2 > 0$ ) and ( $t_1, t_2 < 0$ ). As our system is massive, correlations decay on scale  $1/\Delta$ , where  $\Delta$  is the gap, and therefore most contributions to (26) are only of order  $(t_\perp/\Delta)^2$ . There is, however, one important exception: if a domain wall is created at time  $t_1 < 0$  and position  $x_1$  with  $x \ll x_1 \ll y$  and destroyed at time  $t_2 > 0$ , the order parameter changes by  $e^{\pm i\gamma\beta}$  as at time  $t = 0$  a domain wall is enclosed between  $x$  and  $y$ . Here we assume that  $|x - y|$ ,  $|x_1 - x|$  and  $|x_1 - y|$  are much larger than the correlation length  $\xi \sim 1/\Delta$ . We therefore obtain (up to corrections of order  $\xi/|x - y|$ )

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{+\infty} dx_1 dt_1 dx_2 dt_2 (e^{\pm i\gamma\beta\theta(x-x_1)} e^{\mp i\gamma\beta\theta(y-x_2)} - 1) \times \\ & \quad \langle e^{i\gamma(\Phi_{s,0}(x) - \Phi_{s,0}(y))} \rangle_0 \langle h_\perp(x_1, t_1) h_\perp(x_2, t_2) \rangle \\ & \quad \approx \mu_{SO}^2 |x - y| (\cos(\gamma\beta) - 1) \times \\ & \quad \int_{-\infty}^0 dt_1 \int_0^\infty dt_2 \int_{-\infty}^\infty dx' \langle h_\perp(x', t_1) h_\perp(0, t_2) \rangle \\ & \quad \sim \mu_{SO}^2 \frac{|x - y|}{a} z (\cos(\gamma\beta) - 1) \left( \frac{t_\perp}{\Delta} \right)^2 \left( \frac{\xi}{a} \right)^{1-2\eta} \end{aligned} \quad (27)$$

where  $\mu_{SO}^2$  is the order parameter in absence of the perturbation,  $a$  is the lattice spacing,  $z$  the number of neighboring chains and  $\eta$  is the scaling dimension of  $h_\perp$ . The last factor  $(\xi/a)^{1-2\eta}$  is only of relevance in the asymptotic regime  $|x - y| \gg \xi \gg a$ , i.e. for a gap  $\Delta$  small compared to the band width.

In the cases of the transverse-field Ising model with  $J = 0$  and the non-interacting band insulator, we have shown that the linear correction (27) resums to an exponential. Physically, it is clear that this also will happen for the arbitrary systems considered above: The finite  $t_\perp$  induces a finite density of domain walls and the probability of having no domain walls between  $x$  and  $y$  is therefore

exponentially small. As the non-local string correlations are destroyed by domain walls, the string order vanishes exponentially,

$$SO \approx \mu_{SO}^2 e^{-\alpha|x-y|} \quad (28)$$

where  $\alpha$  can be read off from Eq. (27) in the limit of small  $t_\perp \ll \Delta$ .

In Fig. 3, we compare Eq. (27) with the exact calculation available for the weakly coupled band insulators ( $\eta = 1$ ,  $\beta = \sqrt{\pi/2}$ ). We find nice agreement with the expected  $\alpha \approx 1/\Delta^2$  and  $\alpha \approx 1/\Delta$  behavior both deep in the gapped phase, where  $\xi \approx a$  (left upper panel of Fig. 3) and close to the critical point where  $\xi \gg a$  (right upper panel of Fig. 3). Also the angular dependence is very well described by the factor  $(1 - \cos(\gamma\beta))$  (see lower panel of Fig. 3).

For the case of the quantum Ising chain in transverse field, the  $J = 0$  case has already been considered in Eq. (8). We can now use our result to predict the leading behavior of the decay exponent for  $B \gtrsim J$ . As the scaling dimension for the  $S_z S_z$  coupling between neighboring chains is given<sup>10</sup> by  $\eta = 1/4$ , it follows from Eq. (27) that the string order decays as

$$SO(x - y) \approx \mu_{SO}^2 e^{-\frac{cJ^2|x-y|}{a\Delta^{5/2}/\sqrt{J}}} \quad (29)$$

where  $c$  is a dimensionless constant of order 1.

From Eq. (27) we can directly infer the conditions of stability of the string order. In fact, if the relation

$$\gamma\beta = 2\pi n \quad (30)$$

holds, the order parameter commutes with  $h_\perp$  and remains finite. For example, in the case of the band insulator, we have  $\beta = \sqrt{\pi/2}$  and the string order is formally stable for  $\bar{\gamma} = 4\pi n$  in Eq. (24). However, in this case the operator  $\exp\left(i\bar{\gamma} \sum_{l=i+1}^{j-1} S_{1,l}^z + S_{2,l}^z\right)$  becomes trivially the identity and the correlation function  $\langle S_{j1}^z \exp\left(i\bar{\gamma} \sum_{l=i+1}^{j-1} S_{1,l}^z + S_{2,l}^z\right) S_{i1}^z \rangle = \langle S_{j1}^z S_{i1}^z \rangle$  decays exponentially even for the purely one-dimensional model (see Fig. 2). Therefore no stable order parameter exists for a band insulator (or the Haldane chain).

As a consistency check, we now analyze a case where the order is purely local and therefore stable with respect to small perturbations. For example, spinless Fermions form a charge density wave for sufficiently strong interactions (of sufficient long range). Within bosonization such a system is described by<sup>18</sup>

$$H = \int dx \frac{v_F}{2} \left[ \frac{1}{K} (\partial_x \Theta)^2 + K (\partial_x \Phi)^2 \right] + g \cos(\sqrt{16\pi} \Phi) \quad (31)$$

and for  $K > 1/2$  the cosine term becomes relevant locking the  $\Phi$  field, which implies a spontaneous breaking of the translational invariance for the underlying lattice model. Equivalently, (31) describes the physics of the

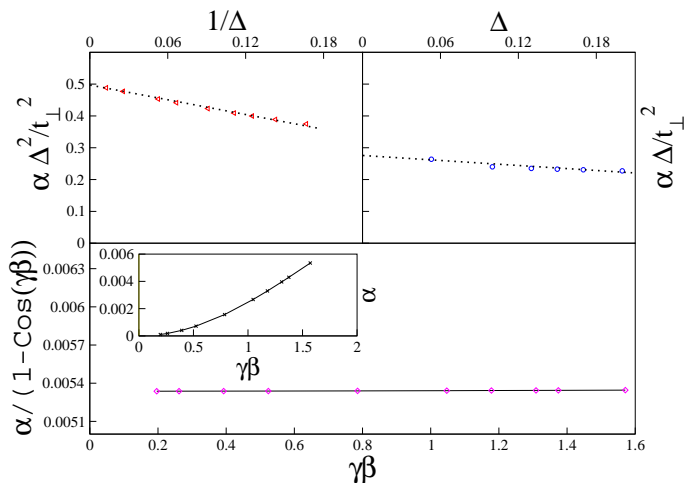


FIG. 3: (Color online) We test the different limits of Eq. (27) in the case of an array of band insulators (the Hamiltonian is defined in Eqns. (17,18)) where we can calculate the SO parameters exactly ( $SO_{\text{odd/even}} \sim e^{-\alpha|i-j|}$ ). Upper left panel:  $\alpha$  multiplied by the ratio  $\Delta^2/t_\perp^2$  versus  $1/\Delta$  in the limit of  $\Delta$  much bigger than the band width ( $\xi \approx a$ ) for the set of parameters  $t_1 = 0$ ,  $t_2 = 0$ ,  $t_R = -1$ ,  $t_D = [-80, -6]$ ,  $t_\perp = -0.2$ . The corrections to Eq. (27) are of order  $1/\Delta$ . Upper right panel:  $\alpha$  multiplied by the ratio  $\Delta/t_\perp^2$  versus  $1/\Delta$  in the scaling limit ( $\xi \gg a$ ) for the set of parameters  $t_1 = 0$ ,  $t_2 = 0$ ,  $t_R = -1$ ,  $t_D = [-1.2, -1.05]$ ,  $t_\perp = -0.02$ . The corrections to Eq. (27) are of order  $\Delta$ . Lower panel:  $\alpha$  divided by the predicted angular dependence  $(1 - \cos(\gamma\beta))$  for the set of parameters  $t_1 = 0$ ,  $t_2 = 0$ ,  $t_R = -1$ ,  $t_D = -4$ ,  $t_\perp = -0.4$  ( $\beta = \sqrt{\pi/2}$  is fixed by the low energy expression of the inter-chain coupling and we vary  $\gamma$  in Eq. (22)). Lower panel inset: the decay exponent versus  $\gamma\beta$  for the same set of parameters.

XXZ spin-1/2 chain where the condition  $K < \frac{1}{2}$  translates to  $J_z > J_{xy}$ . In this case, long-ranged Néel order develops in the ground state. In both cases the relevant order parameter, the staggered component of the Fermionic density  $\Psi_R^\dagger \Psi_L$ , is local. Within bosonization it can be extracted from  $\lim_{|x-y| \rightarrow \infty} \langle e^{i2\sqrt{\pi}(\Phi(x) - \Phi(y))} \rangle$ . A perturbation due to hopping to the neighboring chain or due to a spin-flip is proportional to  $e^{\pm i\sqrt{\pi}\Theta}$ . This implies  $\gamma = 2\sqrt{\pi}$ ,  $\beta = \sqrt{\pi}$  and therefore  $\gamma\beta = 2\pi$ , fulfilling the condition of stability (see Eq. (30)): the charge density wave or the Néel order are “true” local orders in a gapped system which are stable with respect to small perturbations.

In conclusion, we have shown that the string order of gapped one-dimensional systems is very fragile: an arbi-

trarily small coupling to neighboring chains or ladders is sufficient for its vanishing. This has to be contrasted with the behavior of essentially all other ground state properties which are minimally affected. The finite energy gap protects them such that a critical coupling is needed to induce a quantum phase transition.

One may think about possible generalizations of the string order parameter for higher dimensional systems. Indeed, for two coupled spin-1 chains, Todo<sup>17</sup> suggested a generalization of the string-order parameter (also used in Ref. [19] to characterize the phases of frustrated spin-1 chains)

$$\tilde{SO}_2 = \lim_{|i-j| \rightarrow \infty} S_{1,i}^z S_{2,i}^z e^{\sum_{l=i+1}^{j-1} (S_{l,1}^z + S_{l,2}^z)} S_{1,j}^z S_{2,j}^z \quad (32)$$

where  $S_{x,y}$  of Eq. (2) now includes the spins of both chains. The stability of  $\tilde{SO}_2$  comes from the fact that all non-local commutation relations of the string-order and the inter-chain coupling vanish as  $\sum S_{\alpha i}^z$  commutes with  $H_\perp$ . In a completely analogous way, one can define for any finite number of chains  $N$  a generalized order parameter  $\tilde{SO}_N$  (with or without boundary spins) that is non-zero. However, a generalization of the above formula for a two- or three-dimensional system is not possible. First,  $SO_N$  vanishes exponentially even in the *absence* of any coupling between the chains. Second, if the string of Eq. (2) is generalized to a square, a cube or any other finite subset of spins, this will immediately lead again to non-local commutation due the presence of ‘dangling singlets’ at the boundaries of such structures. As for dimensions  $d > 1$  the surface of any non-local structure of infinite extension is infinite, we conclude that no direct extensions of string order to higher dimensional systems can be devised and generalized string orders will decay as  $e^{-\alpha A}$  where  $A$  is the area of the boundary<sup>20</sup>.

An interesting question for the future is the investigation of the stability of various types of topological order. For example, Senthil and Fisher<sup>9</sup> have shown that the ground state degeneracy of the deconfined phase of a two-dimensional  $Z_2$  gauge theory is stable with respect to a small inter-layer coupling. Finally, a related problem is the stability of various types of entanglement measures<sup>21,22</sup>.

We acknowledge useful discussions with J.I. Cirac, M. Garst, R. Moessner, E. Müller-Hartmann, A.A. Nersisyan, A. Schadschneider, A.M. Tsvelik and J. Zaanen, J. Zittartz and, especially, G.I. Japaridze. We thank for financial support of the DFG under SFB 608.

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